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THIS JOURNAL IS DEDICATED TO THE FOLLOWING AIMS: (1) Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values. (2) To supply an additional medium for the publication of expository mathematical articles. (3) To promote more scientific methods of teaching mathematics. (4) To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Every paper on technical mathematics offered for publication should be submitted (with enough enclosed postage to cover two two-way transmissions) to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer. If approved for publication, the Committee will forward it to the Editor and Manager at Baton Rouge, who will notify the writer of its acceptance for publication. If the paper is not approved the Committee will so notify the Editor and Manager, who will inform the writer accordingly.

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GREMLINS

These remarks have as their genesis the belief that this *is* the best of all possible worlds. Not that this obviates the right of an individual to be dissatisfied with his lot nor the opportunity to do something about it. This belief is not exactly an easy one to cling to in view of the seeming fragility of man-made plans. Whatever underlying principles there may be, the role of the individual is purely functional.

We are all resigned, for the duration at least, to the reading of dull articles on "Mathematics and War", etc. However, the apparition of voluminous papers on "Mathematics AFTER the War" is enough to render obsequious even the stout hearted.

While chewing our fingernails we hear that the Gremlins will come out of their caves shortly after the armistice and start enhancing the exalted position which has been thrust on mathematics by virtue of the emergency.

Is not all this concern unwarranted?

Even the army has a traditional re-organization period after cessation of hostilities during which time attempts are made to evaluate lessons recently learned in the theory and practice of warfare. Surely no one will begrudge the Gremlins an opportunity to return mathematics to its fitting and proper place in our esteemed secondary and collegiate educational systems. In fact, so gratified are the Army and Navy officials with the products of said systems that they are even now taking over bodily most of the institutions of higher learning. We hope, at that now indefinite date, when hostilities cease that the Generals and Admirals will still have some small measure of influence on the educational trends of our country, though it is hardly likely they can mitigate blows from the Gremlins more than temporarily.

The Gremlins will argue that "emergency" implies abnormality, hence to judge the results of our educational systems by present requirements is unfair. Even a Gremlin ought to be able to predict by an empirical study of history the recurrence of unusual circumstances of one kind or another. Grant further that most forms of human endeavor which are not completely checked during such periods

(CONCLUDED ON PAGE 247)

Building Triangles with Integers

By L. V. ROBINSON
University of Mississippi

Doubtless many students of undergraduate mathematics are still impressed and perhaps often amazed—as was the writer at that stage in his mathematical studies—by the frequency of the occurrence of whole numbers among the answers to problems usually found in the ordinary text-book. This latter fact is especially noticeable in the case of problems involving right triangles. The critical student who is aware of the fact that integers constitute but a negligible fraction of the whole class of numbers is at once convinced, however, that such is no mere coincidence and suspects that there must be some subtle technique by which such problems are set up. To the dull and uncritical student, on the other hand, the misconception may be further exaggerated that there are few numbers which indeed are not integers; for to him *number* is apt to mean simply *whole* number.

What then is the technique back of such problems? How can that technique be generalized and applied to triangles which are not right triangles? For all those who derive some fascination at the manner in which triangles may be found such that their parts are always integers, the following considerations are therefore presented: to find triangles such that, in addition to the three sides being integers, (1) the radius of the inscribed circle is an integer; (2) the radius of the circumscribed circle is an integer; (3) the radii of the inscribed and of the circumscribed circles are integers when two solutions exist—that is, when two sides and an angle opposite one side of the triangle are given; and (4) the triangle is a right triangle.

Now, to consider the first type of problem, suppose for the moment that the radius of the inscribed circle is some integer r; and let the sides a, b, and c be defined through the equations,

$$a=s-p,$$

$$b=s-q,$$

$$c=s-t,$$

where s is the semi-perimeter of the triangle. It follows therefore that

$$s=p+q+t,$$

and $$r^2=\frac{pqt}{p+q+t}.$$

Then, solving this equation for t, we shall have

$$t=\frac{(p+q)r^2}{pq-r^2}.$$

By means cf the last equation, it is now possible to eliminate t and to reduce the sides and the functions of the angles opposite them to functions of p, q, and r, all of which hereafter we shall take to be integers. Thus,

$$a=\frac{p(q^2+r^2)}{pq-r^2}, \qquad b=\frac{q(p^2+r^2)}{pq-r^2},$$

$$c=p+q, \qquad s=\frac{pq(p+q)}{pq-r^2}.$$

Multiplying the above values of a, b, c, and s by $pq-r^2$, these four quantities thereby become integers provided p, q, and r are integers. It should be pointed out, however, that r is no longer the radius of the inscribed circle, the expression for which hereafter will be R. Moreover, in order to exclude the common factor of 2 among the expressions for the three sides, we may here impose the condition that among the integers p, q, and r, two shall be odd and the other even. We assume also that $pq-r^2>0$. Thus we have

$$(1)\qquad \begin{cases} a=p(q^2+r^2), \\ b=q(p^2+r^2), \\ c=(p+q)(pq-r^2), \\ s=pq(p+q), \\ R=r(pq-r^2), \\ \tan\tfrac{1}{2}A=\dfrac{r}{p}, \\ \tan\tfrac{1}{2}B=\dfrac{r}{q}, \\ \tan\tfrac{1}{2}C=\dfrac{pq-r^2}{(p+q)r}. \end{cases}$$

To these equations we may also add the area, $K = Rs$, which is likewise an integer whose value is

$$K = pqr(p+q)(pq-r^2).$$

Problems belonging to the second category can now be solved by noting that the radius of the circumscribed circle is

$$P = \frac{abc}{4K};$$

or, in terms of p, q, and r,

$$P = \frac{(p^2+r^2)(q^2+r^2)}{4r}.$$

Hence, in addition to the three sides and the radius of the inscribed circle being representable by integers, it is possible also to make the radius of the circumscribed circle likewise an integer. Except in the case where r is even, and in accordance with the conditions imposed above, all the quantities a, b, c R, and P become integers by multiplying the above expressions for them by r. If r is even and p and q are odd, however, then the above quantities must be multiplied by $4r$ to make integers of all of them.

Turning attention now to the problems belonging to the third category, suppose that the given sides are a and b and that the given angle is A—or in the analysis which follows, A_1, say. There are two solutions for such problems when $b = a$. Moreover, if A_1, B_1, and C_1 are the angles for one solution, then for the other,

$$A_2 = A_1, \quad B_2 = 180° - B_1, \quad C_2 = B_1 - A_1,$$

whence

$$\tan \tfrac{1}{2}B_2 = \tan (90° - \tfrac{1}{2}B_1) = \cot B_1,$$

$$\tan \tfrac{1}{2}C_2 = \frac{\tan \tfrac{1}{2}B_1 - \tan \tfrac{1}{2}A_1}{1+\tan \tfrac{1}{2}B_1 \tan \tfrac{1}{2}A_1}.$$

From equation (1) it therefore follows that

$$\tan \tfrac{1}{2}A_2 = \frac{r}{p},$$

$$\tan \tfrac{1}{2}B_2 = \frac{q}{r},$$

$$\tan \tfrac{1}{2}C_2 = \frac{r(p-q)}{pq+r^2}.$$

Assuming now that in the equation,

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

a, b, and A are known and solving for c, we shall have

$$c = b\left(\cos A \pm \sqrt{\frac{a^2}{b^2} - \sin^2 A} \right).$$

The required expressions for $\sin A$ and $\cos A$ in terms of p, q, and r can then be had from the identities,

$$\sin A = \frac{2 \tan \frac{1}{2}A}{1 + \tan^2 \frac{1}{2}A},$$

$$\cos A = \frac{1 - \tan^2 \frac{1}{2}A}{1 + \tan^2 \frac{1}{2}A},$$

and

$$\tan \tfrac{1}{2}A = \frac{r}{p}.$$

Hence, it follows that

$$\sin A = \frac{2pr}{p^2 + r^2},$$

$$\cos A = \frac{p^2 - r^2}{p^2 + r^2}.$$

Substituting these values for $\sin A$ and $\cos A$ and the values of a and b from equations (1) into the above expression for c, we shall have

$$c = q(p^2 + r^2)\left[\frac{p^2 - r^2}{p^2 + r^2} \pm \sqrt{\frac{p^2(q^2 + r^2)^2}{q^2(p^2 + r^2)^2} - \frac{4p^2r^2}{(p^2 + r^2)^2}} \right]$$

$$= (p \pm q)(pq \mp r^2);$$

or

$$c_1 = (p + q)(pq - r^2),$$

$$c_2 = (p - q)(pq + r^2).$$

Hence, whatever values of the integers p, q, and r are chosen to make the sides a and b integers, and through the equation

$$\tan \tfrac{1}{2}A = \frac{r}{p},$$

to specify the angle A, then it must follow that in both cases c is also represented by an integer.

For the second solution, we shall have therefore

$$
(2)\qquad
\begin{cases}
A_2 = A_1, \\
B_2 = 180^\circ - B_1, \\
C_2 = B_1 - A_1, \\
\tan \tfrac{1}{2}A_2 = \dfrac{r}{p}, \\
\tan \tfrac{1}{2}B_2 = \dfrac{q}{r}, \\
\tan \tfrac{1}{2}C_2 = \dfrac{r(p-q)}{pq+r^2}, \\
a_2 = p(q^2+r^2), \\
b_2 = q(p^2+r^2), \\
c_2 = (p-q)(pq+r^2), \\
s_2 = p(pq+r^2), \\
R_2 = qr(p-q), \\
P_2 = \dfrac{(p^2+r^2)(q^2+r^2)}{4r}, \\
K_2 = pqr(p-q)(pq+r^2).
\end{cases}
$$

As before, the radius of the circumscribed circle likewise becomes an integer if the above expression for the sides, the semi-perimeter, and the radius of the inscribed circle be multiplied by the factor of r or $4r$ as explained above. The area K_2 is, of course, multiplied then by r^2 or $(4r)^2$ as the case may be.

A comparison between the two solutions may now be enlightening, as well as of some interest. For instance, the general expressions for c and K are

$$c = (p \pm q)(pq \mp r^2),$$

$$K = pqr(p \mp q)(pq \mp r^2).$$

Moreover, $$R = pqr - r(r^2, \text{ or } q^2),$$

and P is the same in both cases. Both solutions exist, however, only when $p > q$, as is obvious from the above equations. There is, more-

over, but one solution when $p \leqq q$; and the two solutions coincide to give a right triangle when $p > q$ and $q = r$.

Aside from the case just treated, there is no ambiguity in setting up other types of problems involving, for instance: (a) two sides and the included angle, and (b) one side and two angles. The limitations imposed on the angles are those specified in writing the tangents of half the angles, of course.

Now, for the problem belonging to the fourth category, it is apparent that $B = 90°$ when $q = r$. Factoring out the unnecessary factor of q and imposing the condition that of the two quantities, p and q, one is even and the other odd, we shall have

$$(3) \qquad \begin{cases} a = 2pq, \\ b = p^2 + q^2, \\ c = p^2 - q^2, \\ R = q(p - q), \\ P = \frac{1}{2}q(p^2 + q^2), \end{cases}$$

all of which are then prime numbers unless p and q have some odd integers for a common factor. Furthermore, it is apparent that P is an integer when q is even and p odd.

The first three of equations (3) are perhaps the first intimations a student gets of the important field of mathematics pertaining to the theory of numbers. If he is impressed by the fact that this is but a special case of the more general one outlined above, he may be interested in pursuing his studies further. He may then be referred, for instance, to an article by Lehmer in *Annals of Mathematics*, second series, Vol. I, pp. 97-102 for a fuller treatment of the problem.

A Pursuit Problem

By CLARENCE M. CREWS
New York, New York

If a man runs at constant speed from O to B on a straight line and a bull runs at constant speed from a point A, not on OB, directly toward the man, what is the length of the path of the bull if he catches the man at B? The purpose of this paper is to establish the following *Theorem.* The length σ of the path of the bull in the above problem may be obtained by producing OB to K so that $BK=OB$, joining A to K, then

$$\sigma=\tfrac{1}{2}(OA+AK).$$

It may be that this result is not new, but it is hoped it will be of interest to the readers of this Magazine.

It is convenient to discuss three cases. Let the coordinates of A be (x_0,y_0).

Case 1. $x_0<0,\quad y_0>0$

Case 2. $x_0=0.\quad y_0>0$

Case 3. $x_0>0.\quad y_0>0.$

There is no loss of generality in assuming $y_0>0$.

If we choose O as the origin of coordinates, $M(ht,0)$ as the position of the man at time t, $B(b,0)$ the end of the man's path, $P(x,y)$ the position of the bull at time t, s the distance covered by the bull between A and P, c the speed of the bull, then

$$\frac{dy}{dx}=\frac{y}{x-ht} \tag{1}$$

$$\frac{ds}{dt}=c. \tag{2}$$

Case 1. Equations (1), (2) yield after elimination of t

$$\frac{y\,y''}{(y')^2}\quad\frac{c}{\sqrt{1+(y')^2}}=h \tag{3}$$

where $y'=dy/dx$, $y''=d^2y/dx^2$. Introduction of $p=y'$ as a new variable, with y as the independent variable gives

$$\frac{h}{c}\,\frac{dy}{y}=\frac{dp}{p\sqrt{1+p^2}}\,. \tag{4}$$

If $p=\tan\Theta$, $\Theta=\arctan p$ (principal determination), (4) gives

$$x=x_0+\frac{y_0}{2}\left[\frac{\csc\Theta_0+\cot\Theta_0}{1-(h/c)}\left\{\left(\frac{y}{y_0}\right)^{1-h/c}-1\right\}\right. \tag{5}$$
$$\left.-\frac{\csc\Theta_0+\cot\Theta_0}{1+(h/c)}\left\{\left(\frac{y}{y_0}\right)^{1+(h/c)}-1\right\}\right]$$

where
$$\tan\Theta_0=\frac{y_0}{x_0}\,.$$

The ratio $u=c/h$ is determined by the fact that $B(b,0)$ lies on the graph of (5). Thus

$$u=\frac{\sqrt{x_0^2+y_0^2}+\sqrt{x_0^2+y_0^2+4b(b-x_0)}}{2b}\,. \tag{6}$$

The length of the path of the bull is, therefore,

$$\sigma=bu=\tfrac{1}{2}[OA+AK]. \tag{7}$$

Case 2. A similar calculation shows that

$$x=\frac{y_0}{2}\left[\frac{(y/y_0)^{1+(h/c)}-1}{1+(h/c)}-\frac{(y/y_0)^{1-(h/c)}-1}{1-(h/c)}\right] \tag{8}$$

with

$$bu^2-y_0u-b=0 \tag{9}$$

$$\sigma=bu=\frac{y_0+\sqrt{y_0^2+4b^2}}{2}=\tfrac{1}{2}[OA+AK]. \tag{10}$$

Case 3. At A, $ds/dx<0$, so

$$\frac{h}{c}\,\frac{dy}{y}=-\frac{dp}{p\sqrt{1+p^2}}=-\frac{d\Theta}{\sin\Theta} \tag{11}$$

$ds/dx<0$ until $dx/dy=\cot\Theta=0$ at $C(x_1,y_1)$, where

$$y_1=y_0\left[\frac{\sqrt{x_0^2+y_0^2}-x_0}{y_0}\right]\frac{c}{h}\,. \tag{12}$$

Beyond $C(x_1,y_1)$ the calculations follow equation (4), with the result that

$$bu^2-\sqrt{x_0^2+y_0^2}u-b+x_0=0 \qquad \text{whence} \tag{13}$$

$$\sigma=bu=\tfrac{1}{2}(OA+AK). \tag{14}$$

If $\sigma=a=$semi-major axis of an ellipse, the locus of points $A(x,y)$ of departure for the bull is an ellipse if the point O is at one of the foci and B is the center of the ellipse.

GREMLINS

(CONCLUDED FROM PAGE 238)

become greatly accelerated. Then it is not quite clear that the least wasteful methods of providing a technical education at such unusual times might not also in normal times provide the least wasteful methods of accomplishing some of the high sounding objectives as "preparation for and participation in living." Such a training might at least endow the educatee with some skill or knowledge which could be used to establish beyond any shadow of doubt the fact that the individual has been to school. It is conceivable that any kind of disciplinary training for the mind might tend to reduce the incidence of illiterary. However, it must be understood that such a wild conjecture as this is one for which we alone must assume full responsibility for the Gremlins have long suspected that such disciplinary training is not only valueless but also is actually harmful to the individual and society in some strange and mysterious way.

By this time our Gremlin friends are purple with rage. But wait—they have an out! We recall the story of the traveller who stopped at a rural inn, and after a tasty meal, inquired the cost. "Fifty cents," replied the maid. "Fifty cents" ! exclaimed the man. "Certainly you lose money on such a fine meal." "Sure, we do a bit," replied the maid, "But we make up for it by serving a lot of them." With the same impeccable logic our Gremlin friends have convinced everyone that all is well and that after the war, in the words of our friends, we may dare return and educate the young.

O. G. HARROLD.
Louisiana State University.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON and A. W. RICHESON

Arithmetic According to Cocker

By E. R. SLEIGHT
Albion College

During the early years of the development of arithmetic in England, two schools of arithmeticians existed,—the commercial and the demonstrative. De Morgan states that "It was never the habit of writers of text books on the subject to prove their rules; only to test the correctness of an operation by a reverse process, casting out the nines, and others similar to these. As soon as attention was fairly diverted to Arithmetic for Commercial purposes, such explanations as had been handed down from the writers of the 16th century began to disappear, and finally became extinct in Cocker." From this time began the finished school of teachers, whose pupils asked, when a particular example disturbed them, what rule is it? De Morgan further says it is "to the Commercial school of Arithmeticians we owe the destruction of demonstrative arithmetic in England, or rather the prevention of its growth."

Very little is known about Edward Cocker. He was born in 1631, and in 1657 he was living "on the south side of St. Paul's churchyard where he taught the art of writing and arithmetic in an extraordinary manner." He also collected books, and in his public schools for writing and arithmetic where he took in boarders he had a large library of manuscripts and printed books in various languages. He wrote three or four books on arithmetic and penmanship, but it was his "Arithmetic, being a Plain and Easy Method, edited by John Hopkins" in 1678 that gave him a place among the really great men of his day. No arithmetic in the English language has had as many editions as Cocker's and only Record's *Ground of Artes* was used over as long a period.* "Approximately one hundred editions appeared in the British Isles, but strange to relate only one edition appeared in Amer-

*Karpinski's *History of Arithmetic.*

ica." Cocker dropped old terminology, condensed the presentation of many topics, dropped all mention of computation by counters, and introduced many lists of problems.

That there was a keen appetite for a good arithmetic book at this time is shown by the number of editions, as well as the number of years it continued in use. It is difficult to account for its proverbial reputation of accuracy, as the phrase, "According to Cocker" indicates. It did, however, teach arithmetic by a simple method, "suited to the meanest capacity." The rules were stated concisely, without proof, each rule illustrated by an example worked out in detail. Other examples for practice are given in the text, but always following the rule by which they are to be worked. These may be some of the reasons why it was a "best seller" of the times.

Many historians attribute the classical statement "According to Cocker," to his ability as a teacher of Arithmetic. I am inclined, however, to believe differently. In the city library at Edinburgh, Scotland, there is a paper covered copy book entitled "Penmanship according to Cocker." In the preface, Cocker deplores the fact that there is so little interest in Arithmetic that he found it necessary to turn to penmanship for a living. I feel quite sure that originally "According to Cocker" was attributed to him, not because of his arithmetic, but because of his penmanship.

As an introduction to the text we find a preface by Cocker in which he addresses the public, the professors, the ingenious off springs, and the numerists of this Vaporing age. "By the sacred Influence of Divine Providence; I have been Instrumental to the benefit of many, by virtue of those useful Arts, Writing and Engraving: and do now by the same alacracity cast this my arithmetical mite into the Public Treasury, beseaching the Almighty to grant the like blessing to these as to my former Labors.

"I have often desired by my intimate Friends to publish something on this Subject, who in pleasing freedom have signified to me that they expected it would be extraordinary. How far I have answered their expectations, I know not; but this I know I have designed this work by all means possible within the Circumference of my Capacity, endeavoring to render it useful to all those whose occasions shall induce them to make use of Numbers. Knowing that Merchandise is the life of the public, I do profess that the numerous concerns of the honoured Merchants first possessed my considerations.

"Secondly, For your service, most excellent Professors, whose Understandings soar to the sublimity of the Theory and Practice of this Noble Science, was this Arithmetical Tractate composed, in order

that you may take occasion to reserve your precious moments for more precious affairs.

"Thirdly, For you, the ingenious Off-springs of happy Parents, who will willingly pay the full price of industry, for you I say was this work composed, which may prove the pleasure of your Youth and the Glory of your age.

"Lastly, for you the pretended Numerists of this vapouring Age, who are more disingenously witty to propound unnecessary questions than ingeniously judicious to solve them. For you was this Book composed and published."

Then follows a statement by John Hawkins, the publisher, addressed to the Courteous Readers, "I have had the happiness of an intimate acquaintance with Mr. Cocker in his life time, often solicited him to remember his promise to the world of publishing his Arithmetic, but (for reasons best known to himself) he refused it; and (after his death) the copy falling accidentally into my hands, I thought it not convenient to smother a work of so considerable a moment; not questioning but that it might be as kindly accepted as if it had been printed by his own hands."

The inconsistency in the statements made by Hawkins in his introduction to the "Courteous Readers," and those found in the preface by Cocker has lead to considerable controversy over the authorship of this Arithmetic. De Morgan believed that the book was never published during the life of Cocker, but rather that John Hawkins was the real author, aided, perhaps by some of Cocker's notes. It seems to me that the inconsistencies may well be harmonized in that the text was evidently in manuscript form before it came into the hands of Hawkins, and the preface as stated by Cocker was a part of the original manuscript. David Eugene Smith in his Rara Arithmetica, makes the statement that "the world is much better off because of these books, whoever wrote them."

An analysis of Cocker's Arithmetic shows that the first chapter is devoted to "Notation of Numbers", and consists of various types and sequences of numbers; for as is stated "multitude of Numbers is the subject of Arithmetic, and if so, then the first principles and fundamentals must have like definitions." Among the disputed concepts of the day, he points out that "most authors maintain that unit is the beginning of number, and itself no number." But the author takes the position that "*O* is the beginning of number, and hence *unit* is within the bounds of numbers, and is itself a number."

In this first chapter we also find the interesting statement that "Number is either absolute or negative." The term absolute is applied

to integers, but "A negative or broken number, fractional decreasing number, is that which by prefixing a point or prick towards the left its value is decreased by so many units to so many tenths part of anything; and if a point and a cypher be prefixed it will then be so many hundreth parts." Thus he classifies decimals as negative numbers. Fractions are classified as broken numbers, but later in the text they are referred to as fractions. "When a single broken number hath for its denominator a number consisting of a unit in the first place toward the left, and nothing but cyphers from the unit toward the right, it is then aptly and rightly called a decimal fraction. We signify them to be decimals by placing a point, or a prick, before the numerator, rejecting the denominator. Therefore 25/1000 is a decimal, and should stand thus .025."

"In decimal Arithmetic we always imagine, and it would be commodious if it were really so, that all entire units, integers, and things are divided first into 10 equal parts; and these parts so divided we call primes; and, secondly, we divide also each of the primes into 10 equal parts; and every one of these divisions we call seconds. The next division we call thirds, and so on ad infinitum." Logarithms are classed as *borrowed numbers*.

"Concerning Arithmetic itself we find that it is either natural, artificial, analytical, algebraical, lineal, or instrumental. *Natural Arithmetic* is performed by the numbers themselves, while *artificial* is applied when logarithms are used.

"*Analytical Arithmetic* shows how from a thing unknown to find truly that which is sought. *Algebraical Arithmetic* is an obscure and hidden art of accomplishing by numbers, in resolving of hard questions." He also applies the term *Instrumental Arithmetic* when an instrument is used.

"The parts of Arithmetic are Numeration and extraction of roots. Numeration hath four species, viz. addition, subtraction, multiplication and division." The four fundamental operations are discussed at length. In all of the editions I examined it is significant to note that all of these operations are performed precisely as of today, all traces of the *scratch* method in division used by Record, and the *Austrian Method* used by many of the early writers having disappeared.

A chapter on *Comparative Arithmetic* is found in each of the editions, *Comparative Arithmetic* being defined as "that which is wrought by numbers as they are considered to have relation one to another; and this consists either in quantity or in quality. The *relation of numbers in quantity* consists in the *difference* or *rate of reason* that is found betwixt the two numbers propounded; the *difference* being the remainder

found by subtraction; but the *rate of reason* is the quotient of the first number divided by the second."

The *relation of numbers in quality* has reference to the relation of numbers when there are more than two. Under this head are discussed certain sequences, particularly the progressions. Many of the known formulas are written out in full.

The ever present *Rule of Three* and Allegation Medial and Allegation Alternate are present. In the former we find the colon and the double colon used to express single rule direct and single rule inverse.

As one writer expresses it, "The economic and commercial conditions of the period is always expressed in the arithmetics." So here we find many pages devoted to Rules of Practice, Fellowship, the Rule of Barter, Questions in Loss and Gain, Exchange and Equation of Payments.

Fractions have caused difficulty, both to the teacher and to the learner for better than thirty centuries. Perhaps the manner in which fractions are written has something to do with these difficulties. The treatment of fractions is decidedly modern, except that the expression *Vulgar Fractions* is used to distinguish them from decimal fractions.

"How to reduce a vulgar fraction we shall teach under these eight several heads, or rules, following, viz.

1. To reduce a mixed number into an improper fraction.
2. To reduce a whole number into an improper fraction.
3. To reduce an improper fraction into its equivalent whole or mixed number.
4. To reduce a fraction into its lowest term equivalent to the fraction given.
5. To find the value of a fraction in the known parts of coin, weights, and measures.
6. To reduce a compound fraction to a simple one of the same value.
7. To reduce divers fractions having unequal denominators, to fractions of the same value, having equal denominators.
8. To reduce a fraction of one denomination to another of the same value.

For each of these eight reductions there is a rule, followed by several examples, one being worked to illustrate the process. In some cases the rule is brief and clear, as is found in case 1. "The rule is multiply the integer part, or whole number, by the denominator of the fraction, and to this product add the numerator, and that sum place over the denominator for a new number; so this new fraction shall be equal to the mixed number proposed."

In other cases the instructions are long drawn out, and very obscure as to meaning. For reduction (5) the rule is "multiply the numerator by the parts of the next inferior denomination that are equal to an unit of this same denomination with the fraction, then divide that product by the denominator, and the quote (quotient) gives you its value in the same parts you multiplied by; and if anything remain, multiply it by the parts of the next inferior denomination, and divide as before. Do so until you can bring it no lower, and the several quotients will give you the value of the fraction required; and so if anything remain, place it for a numerator over the former denominator."

In addition of fractions, Cocker suggests that the common denominator be the product of all the denominators of the various fractions. "To reduce a fraction to its lowest terms, if the numerator and denominator are even numbers, take half the one and half the other, as often as may be; and when either of them falls out to be odd, then divide them by any odd number that you can discover." Following several examples there appears this statement. "There is yet another way more excellent than the former." Here he outlines the plan of dividing the numerator and denominator by the highest common factor, the factor being found by the Euclidean Algorithm. Clearly this latter plan is taken directly from some other text, as he gives an obscure reference. This is only one of many times that this has been done in the text. This fact caused de Morgan to state that the book is merely a compilation of rules and methods, taken bodily from previous authors.

Concerning multiplication, "If the multiplicand and multiplier are both simple fractions, then multiply the numerators together for a new numerator, and the denominators for a new denominator." Special instructions are given when the fraction is to be multiplied by a mixed number, when a compound fraction is to be multiplied by a mixed number, and when a fraction is to be multiplied by a whole number. Likewise in division of vulgar fractions, there are special instructions for all of the possible cases, the general rule being "multiply the numerator of the dividend by the denominator of the divisor for a new numerator, and the denominator of the dividend by the numerator of the divisor for a new denominator."

The chapters on fractions close with the applications of the rule of three, direct and inverse.

Unlike many of the early arithmetics we find very little use of crosses and other devises "as aids to the learner." About the only exception is to be found in the discussion concerning *single* and *double*

position, or the rule of *false assumption* as it was called in many of the texts of those early days. Cocker refers to this method of procedure as *Negative Arithmetic*, or *The Rule of False*. "The rule of single position is, when at once, viz. by one false position, or feigned number, we find out the true number." To illustrate, suppose a man spends 1/3 of his money for groceries and 1/5 for rent and has \$70 left. How much money had he? Let \$300 represent the original amount. He would then have \$140 left. Then using the proportion $70:140::x:300$, the actual amount is found to be \$150.

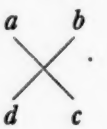
"When two false positions are assumed to give a resolution of the question propounded", then the rule of double position is used. In applying this rule, Cocker made use of the St. Andrew's Cross to help visualize the processes. "When any Question is stated in Double Position, make such a cross as is here found

Then make choice of any number you think may be convenient, placing it at the end of the cross as *a*

then work with this position as though it were the true number sought. Then having found the error as in single position, place it on the side of the cross *d*

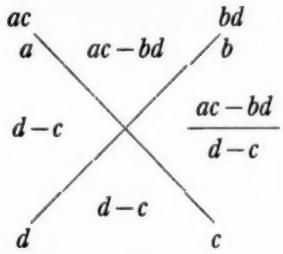
Then make choice of another number, and place it on the cross as *b*

Now find the error and place it on the cross at *c*



Now multiply the errors into the numbers (positions) crosswise, that is the first error into the second position and the second error into the first position, and put each product over its position

Now consider whether the errors were both alike, that is both too large or both too small. If they are both alike, then subtract the lesser product from the greater, and use the remainder as a dividend, and subtract the lesser error from the greater and use this as a divisor. The quotient thus arising will be the answer." For the completed problem the cross could be thus represented

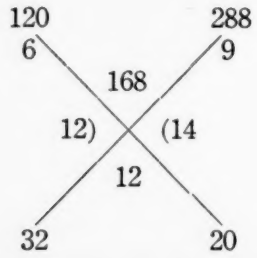


As an illustration, "A, B, and C build a house costing 76 pounds, of which *A* paid a certain sum unknown, B paid as much as A and 10 pounds more and C paid as much as A and B. Now I desire to know each man's share in that charge?

"Having made a cross according to the rule, I come to make a choice of my first position, and here I suppose A paid 6 pounds, which we put on the cross as you see. Then B paid 16 pounds and C paid 22 pounds. The sum of these 3 amounts is 44. The error, therefore is 32, which we place below the 6."

Secondly, suppose A paid 9 pounds, then B would pay 19 pounds, and C 28, a total of 56. The error being 20, we place the 9 and 20 as indicated. Now cross multiply as indicated and place the products

above the 6 and 9 respectively. Following the rule as outlined, 168 and 12 are the differences



of the products and errors respectively. Since both errors are too small, the difference is taken in each case. Then dividing 168 by 12, the result is 14, "which satisfies the problem." If one of the errors is too small and the other too large, then Cocker advises that we divide the sum of the products by the sum of the errors "to obtain the solution." In either case it is the algebraic differences, but as negative numbers had not yet found their way into mathematics, two distinct rules were necessary.

After examining several editions of this famous Arithmetic I am puzzled at its popularity. If we compare definitions and processes as given by Cocker with those of several of the texts just preceding this period we are convinced that very little was gained, and much was lost. For instance defining subtraction according to Recorde, Hodder and Cocker we note,—

Recorde (1532). Subtraction diminisheth a grosse sum by withdrawing of other from it, so that subtraction or Rebating is nothing els, but an arte to withdrawe and abate one sum from another, that the remainder may appeare.

Hodder (1664). Subtraction teacheth to take any lesser number out of a greater, and to know what remains.

Cocker (1678). Subtraction is the taking of a lesser number out of a greater of like kind, whereby to find out a third number, being or declaring the inequality, excess, or difference between the numbers given, or Subtraction is that by which one number is taken out of another number given, to the end that the residue, or remainder may be known, which remainder is also called the rest, Remainder, or difference of the numbers given.

Comparing the definitions for multiplication we discover that Recorde used $3^1/_2$ lines, Hodder $2^1/_2$ and Cocker 11. Several other

predecessors of the author of this *best seller* seemed to know when to stop, but Cocker continued indefinitely. Perhaps de Morgan in his Arithmetical Books is too severe in his criticism, but I feel that he had some foundation for his statement that the reasons for its reputation are "the intrinsic goodness of its processes, in which the book has nothing original; and the systematic puffing with which it was introduced. I am of the opinion that Edward Cocker must have had a great reputation, since a bad book under his name pushed out the good ones."

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Infinity---Its Cause and Cure

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Since Voltaire enunciated the dictum on the conduct of a conversation, it has become quite popular to define your terms before conversing; however, if I could give a precise definition of the concept of infinity, I would mail statements of indebtedness to the amount of an aleph dollars to the theologians and mathematicians the world over for professional services rendered. If paid, I would feel cheated if my immortal "real" being was not given a bonus of a third row-center seat in Allah's Seventh Heaven where the dark-eyed maidens[1] are the feature attraction. Indeed, it is an infallible rule that the address of anyone that would attempt a definition of infinity is one of three: a mathematics department, a religious organization, or a psychopathic ward—any three inmates picked at random will constitute a Trinity of Infinity. My address is a department of mathematics. Infinity is a pink elephant that is the result of intoxication by medieval scholasticism. The purple alligator hallucination is also popular; the inebriant was first brewed by Brouwer[2] with the mathematical adaptation

[1] Paradise is a particularly pleasant place, and the female inhabitants are worthy of careful attention. The following synthesized description is from the revelations of Seventh Heaven as told by Mohammed in the Koran. Paradise is the region of the gardens of perpetual abode where the immortal believing souls, clothed in green silk and wearing gold and silver bracelets, repose on couches adorned with gold and precious stones. Youths which shall continue in their bloom forever, shall go round about to attend them, with golden goblets and breakers of silver and a cup of flowing wine: their heads shall not ache by drinking same, neither shall their reason be disturbed: and with fruits of the sorts which they shall choose, and the flesh of birds which they shall desire. And there shall accompany them the ever-virgin damsels of Paradise, refraining from beholding any besides their spouses, having large black eyes and full round breasts: resembling pearls hidden in their shells. This shall be the reward of the righteous. But for the transgressors is prepared an evil receptacle, namely, hell. Mathematically speaking, hell is the negation of Paradise; the reader and I feel sorry for the sinners roasting in the furnaces of the lower regions.

[2] Brouwer fired the first shot of the world infamous Göttingen-Amsterdam battle. At the present time no Perseus has appeared to decapitate the Medusa heads of mathematical analysis, and the sacred mission entrusted to future generations of mathematicians by Kronecker has not yet been fulfilled. For the ammunition of the first shot see L. E. J. Brouwer, *Intuitionism and Formalism* (Trans. A. Dresden), *Bull. Amer. Math. Soc.*, Vol. 20 (1913-14), pp. 81-96.

of the public transportation systems' motto—There is always room for more.

What is medieval about infinity? Simply this. Most questions on the finite that trouble the research mathematician of today are nothing more or less than modern mathematical paraphrases of "How many angels can sit, or perch,[3] on the head of a pin at one time?"

In addition to the last question, the horns of the dire dilemma of infinity take on many other names; one of the more popular sets is the chicken and egg problem with special emphasis on prior existence. The certain assumptions inherent in the question are called common sense—or quite synonymously, nonsense; one mathematical paraphrase of the assumptions is Peano's fifth postulate for the positive integers;[4] the theological and metaphysical restatement is the princciple of cause and effect. Is the question answered under these assumptions? Oh, indeed yes. The Jabberwockian retort to the problem is either and neither; the mathematical reply has been made by Zermelo by an existence proof of the well ordering of the space of chickens and eggs;[5] the theological answer is that God made God. The reader and I, being the only intelligent people present, strongly suspect that most of the eggs were rotten.

The previous example is not the only one that could be explicitly stated; thinking of a thought is the old standby. Heeding its persistent pleadings, it shall not be disturbed.

It would not be wise to recapitulate the difficulties of transfinite analysis, the consistency of arithmetic, the existence of irrational numbers (i. e., thinking of a thought), and other steeds of divers hues; however, one mount will be ridden.

[3] This heretical form of the question was not popular during the Middle Ages. The standard form was *Ultrum Angelus moveatur de loco ad locum transeundo per medium.* The "correct" answers are: "*Quia forma creata sic subsistens habet esse et non est suum esse, necesse est quod ipsum esse sit receptum et contractum ad terminatum*" (T. Aquinas, *Summa Theologica*, Ia, q. 7, a. 2, and q. 3) and "*Unde esse earum non est absolutum sed receptum, et ideo limitatum et finitum ad capacitatem naturae recipientis*" (T. Aquinas, *De Ente et Essentia*, cap. 6). The question, in its philosophical form, is due to Aristotle (Zeller, *Die Phil. d. Griechen*, II, 2, p. 339, N. 3).

[4] A statement of the postulates and an excellent presentation of the consequences are to be found in C. C. Mac Duffee, *An Introduction to Abstract Algebra*, Chap. I, pp. 1-46. The original memoir is Giuseppe Peano, *Arithmetices Principia*, Turin, 1889.

[5] The opinion of most eminent mathematicians is that Jabberwocky's and Zermelo's answers are equivalent. The reader should take the advice of Buddha and form his own opinion by reading Ernst Zermelo, "*Beweis, dass jede Menge wohlgeordnet werden kann,*" *Math. Ann.*, Vol. 59 (1904), pp. 514-516 and "*Neuer Beweis für die Möglichkeit der Wohlordnung, ibid.*, Vol. 65 (1908), pp. 107-128.

This broncho is known as Gödel's theorem.[6] Gödel, playing the game according to all of the rules, proved that logic is incompletable by recursive means; Quine has extended the theorem to the protosyntactical incompletability of protosyntax. At first glance, it appears that the Inquisition has the Inquisitors; in common sense, it has been shown that a steady diet of logic leads to indigestion. If the meaning isn't too clear, don't worry; the Inquisitors have been reprieved by Carnap with some previously unnoticed interpretations of the rules of the game. Another escape from the Gödel-haunted woodlands of Weir is the elimination from logic of the so-called quasi-equivalences.

All of the above is to set the scene; the soliloquy follows. To prevent the frequent appearances of pink elephants on the wall and purple alligators in the hall, mathematicians should take the pledge.

As a mathematician, I pledge myself to ascertain in a finite number of steps the finite truth values of a finite number of statements about a finite number of elements using a finite number of postulates and definitions.

A great many auditors of this score have complained that they hear the tinnabulations of a Bell; the arranger pleads guilty. My tongue is getting tired—I guess I'll take it out of my cheek.

[6] There exists at least one statement in any logical system such that the theorem is neither true nor false relative to that system. The last statement is Gödel's Theorem in a nutshell. The perfect example of a difficult exercise in notation and concentration is Kurt Gödel, *"Ueber formal unentscheidbare Sätze der 'Principia Mathematica' und verwandter System I," Monatsh. für Math. u. Phys.*, Vol. 38 (1931), pp. 173-198.

The Teachers' Department

Edited by
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Decomposition of Rational Fractions Into Partial Fractions*

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Introduction. Although the question of decomposition of rational fractions into partial fractions is both quite elementary and quite fundamentally important (e. g. in the theory of systematic integration), it has been sadly neglected in American mathematical texts.

It is true that some foreign texts devote more attention to this problem. Yet even the best of these fail to give a completely satisfactory treatment.

Therefore a brief but logically complete exposition of this subject is not out of place.

The problem presents two distinct aspects:

(1) The existence and uniqueness of the decomposition of a rational fraction into partial fractions, and

(2) The determination of the numerators of the partial fractions.

Existence and Uniqueness of Decomposition. Let

$$\frac{f(x)}{F(x)}$$

be a proper rational fraction with real coefficients and in its lowest terms.

The existence and uniqueness of decomposition of

$$\frac{f(x)}{F(x)}$$

*Presented to the Southwestern Section of the Mathematical Association of America at Lubbock, Texas, April 28, 1941.

into partial fractions follows from the following two lemmas:

Lemma I. If $F(x) \equiv (x-\alpha)^n \varphi(x)$, $\varphi(\alpha) \neq 0$, then

$$\frac{f(x)}{F(x)} \equiv \frac{A}{(x-\alpha)^n} + \frac{\psi(x)}{(x-\alpha)^{n-1}\varphi(x)}, \text{ where } A = \frac{f(\alpha)}{\varphi(\alpha)} \neq 0,$$

$\psi(x)$ is unique, and $\psi(x)$ and $\varphi(x)$ are relatively prime.

Proof. We have identically:

$$\frac{f(x)}{F(x)} \equiv \frac{A}{(x-\alpha)^n} + \frac{f(x) - A\varphi(x)}{(x-\alpha)^n \varphi(x)}.$$

Let A be determined by the equality

$$f(\alpha) - A\varphi(\alpha) = 0, \quad \text{i. e. let } A = \frac{f(\alpha)}{\varphi(\alpha)}.$$

This is always possible since $\varphi(\alpha) \neq 0$. Also, since

$$f(\alpha) \neq 0, \quad A \neq 0..$$

With this choice of A, $f(x) - A\varphi(x)$ must contain $x-\alpha$ as a factor:

$$f(x) - A\varphi(x) \equiv (x-\alpha)\psi(x),$$

and

$$\frac{f(x)}{F(x)} \equiv \frac{A}{(x-\alpha)^n} + \frac{\psi(x)}{(x-\alpha)^{n-1}\varphi(x)}.$$

It may be noted that since

$$f(x) \equiv A\varphi(x) + (x-\alpha)\psi(x),$$

$\psi(x)$ and $\varphi(x)$ have no factor in common. Otherwise this factor would divide $f(x)$. This is impossible because

$$\frac{f(x)}{F(x)}$$

is in its lowest terms.

To prove uniqueness, assume:

$$\frac{A_1}{(x-\alpha)^n} + \frac{\psi_1(x)}{(x-\alpha)^{n-1}\varphi(x)} \equiv \frac{A_2}{(x-\alpha)^n} + \frac{\psi_2(x)}{(x-\alpha)^{n-1}\varphi(x)}.$$

Then

$$A_1 - A_2 \equiv \frac{\psi_2(x) - \psi_1(x)}{\varphi(x)} (x-\alpha).$$

Now let $x=\alpha$.

It follows that $A_1=A_2$, and $\psi_1(x)\equiv\psi_2(x)$.

Lemma II. If $F(x)\equiv[(x-a)^2+b^2]^m\varphi(x)$, $\varphi(a+bi)\neq 0$, $b\neq 0$,

then $$\frac{f(x)}{F(x)}\equiv\frac{Ax+B}{[(x-a)^2+b^2]^m}+\frac{\psi(x)}{[(x-a)^2+b^2]^{m-1}\varphi(x)},$$

where A, B, and $\psi(x)$ are uniquely determined, A and B are not both zero, and $\psi(x)$ and $\varphi(x)$ are relatively prime.

Proof. Obviously

$$\frac{f(x)}{F(x)}\equiv\frac{Ax+B}{[(x-a)^2+b^2]^m}+\frac{f(x)-(Ax+B)\varphi(x)}{[(x-a)^2+b^2]^m\varphi(x)}.$$

Let A and B be so determined that

$$f(a+bi)-[A(a+bi)+B]\varphi(a+bi)=0, \quad \text{i. e. so that}$$

$$A(a+bi)+B=\frac{f(a+bi)}{\varphi(a+bi)}=\alpha+\beta i, \text{ say.}$$

This is always possible since $\varphi(a+bi)\neq 0$. But then

$$A=\frac{\beta}{b}, \qquad B=\alpha-\frac{a\beta}{b}.$$

Clearly A and B cannot both be zero, because $f(a+bi)\neq 0$. With this choice of A and B

$$f(x)-(Ax+B)\varphi(x)\equiv(x-a-bi)(x-a+bi)\psi(x)\equiv[(x-a)^2+b^2]\psi(x),$$

and $$\frac{f(x)}{F(x)}\equiv\frac{Ax+B}{[(x-a)^2+b^2]^m}+\frac{\psi(x)}{[(x-a)^2+b^2]^{m-1}\varphi(x)}.$$

As in Lemma I, in this case also $\psi(x)$ and $\varphi(x)$ have no common factor, and the decomposition is unique.

The Main Existence Theorem. Let

$$\frac{f(x)}{F(x)}$$

be a proper rational fraction with real coefficients and in its lowest terms.

If $F(x)\equiv\prod_{k=1}^{p}(x-\alpha_k)^{n_k}\prod_{r=1}^{q}[(x-a)_r)^2+b_r^2]^{m_r}$, then

$$\frac{f(x)}{F(x)} \equiv \sum_{k=1}^{p} \sum_{i=0}^{n_k-1} \frac{A_{ki}}{(x-\alpha_k)^{n_k-i}} + \sum_{r=1}^{q} \sum_{j=0}^{m_r-1} \frac{A_{rj}x+B_{rj}}{[(x-a_r)^2+b_r^2]^{m_r-j}},$$

the decomposition being unique.

This theorem results from the repeated use of Lemmas I and II.

Determination of the Numerators. It is convenient in developing the methods of determination of the numerators of the partial fractions to consider separately two cases:

Case I. The determination of the numerators of partial fractions corresponding to a real linear factor of $F(x)$;

Case II. The determination of the numerators of partial fractions corresponding to a prime quadratic factor of $F(x)$.

It is patently unnecessary to differentiate between the cases when a given factor of $F(x)$ appears raised to the first power or when it is "repeated". A so-called "distinct" factor is simply a "repeated" factor of multiplicity one.

Case I. If $F(x) \equiv (x-\alpha)^n \varphi(x)$, $\varphi(\alpha) \neq 0$, then

$$\frac{f(x)}{F(x)} \equiv \sum_{k=0}^{n-1} \frac{A_k}{(x-\alpha)^{n-k}} + \frac{\psi(x)}{\varphi(x)}.$$

From this

$$\frac{f(x)}{\varphi(x)} \equiv \sum_{k=0}^{n-1} A_k(x-\alpha)^k + (x-\alpha)^n \frac{\psi(x)}{\varphi(x)}.$$

Differentiating both sides of this identity k times and letting $x=\alpha$, we deduce

$$A_k = \frac{1}{k!}\left[\frac{d^k}{dx^k}\ \frac{f(x)}{\varphi(x)}\right]_{x=\alpha}, \quad k=0,1,\cdots,n-1.$$

Case II. If $F(x) \equiv [(x-a)^2+b^2]^n \varphi(x)$, $\varphi(a+bi) \neq 0$, $b \neq 0$, then

$$\frac{f(x)}{F(x)} \equiv \sum_{k=0}^{n-1} \frac{A_k x+B_k}{[(x-a)^2+b^2]^{n-k}} + \frac{\psi(x)}{\varphi(x)},$$

and $$\frac{f(x)}{\varphi(x)} \equiv \sum_{k=0}^{n-1} (A_k x+B_k)[(x-a)^2+b^2]^k + [(x-a)^2+b^2]^n \frac{\psi(x)}{\varphi(x)}.$$

Let $t=(x-a)^2+b^2$. Then

$$\text{(A)} \qquad \frac{f(x)}{F(x)} \equiv \sum_{k=0}^{n-1} (A_k x+B_k)\, t^k + t^n \frac{\psi(x)}{\varphi(x)}.$$

Also $$\frac{d}{dt} \equiv \frac{1}{2(x-a)} \quad \frac{d}{dx},$$

$$\frac{d^2}{dt^2} \equiv \frac{1}{4(x-a)^2} \quad \frac{d^2}{dx^2} - \frac{1}{4(x-a)^3} \quad \frac{d}{dx}, \quad \text{etc.}$$

And by induction:

$$\frac{d^m}{dt^m} \equiv \sum_{r=1}^{m} \frac{(-1)^{r+1}\, {}_mN_r}{2^m (x-a)^{m+r-1}} \quad \frac{d^{m-r+1}}{dx^{m-r+1}},$$

where the numbers ${}_mN_r$ are defined by:

(1) ${}_mN_1=1$ for all $m \geqslant 1$.

(2) ${}_mN_r=0$ for $r>m$.

(3) ${}_mN_r = {}_{m-1}N_r + (m+r-3)\, {}_{m-1}N_{r-1}$.

We now operate on both sides of (A) with $\frac{d^m}{dt^m}$, $m=0,\ 1,\ 2\cdots$, $n-1$, and let $t=0$, i. e. $x=a+bi$.

Observing that for $t=0$,

$$\frac{d^m}{dt^m}\left[(A_k x+B_k)\, t^k\right] \equiv \sum_{r=0}^{m} {}_mC_r \frac{d^{m-r}}{dt^{m-r}}(A_k x+B_k) \cdot \frac{d^r}{dt^r} t^k$$

$$=0 \quad \text{for} \quad k>m,$$

$$=\left[A_m(a+bi)+B_m\right] \cdot m! \quad \text{for} \quad k=m,$$

and

$$=\left[{}_mC_k \frac{d^{m-k}}{dt^{m-k}}(A_k x+B_k) \cdot k!\right]_{x=a+bi}$$

$$=(-1)^{m-k+1} \frac{{}_mC_k \cdot {}_{m-k}N_{m-k} \cdot k!\, A_k}{2^{m-k}(bi)^{2(m-k)-1}}, \quad \text{for} \quad k<m,$$

while $$\frac{d^m}{dt^m}\left[t^n \frac{\psi(x)}{\varphi(x)}\right]=0,$$

we deduce from (A):

$$\text{(B)} \qquad \frac{f(a+bi)}{\varphi(a+bi)} = A_0(a+bi)+B_0, \quad \text{and for} \quad m \geqslant 1,$$

$$(C)\quad \sum_{r=1}^{m} \frac{(-1)^{r+1}{}_mN_r}{2^m(bi)^{m+r-1}} \left[\frac{d^{m-r+1}}{dx^{m-r+1}} \frac{f(x)}{\varphi(x)}\right]_{x=a+bi}$$

$$=\sum_{k=0}^{m-1} \frac{(-1)^{m-k+1}{}_mC_k \cdot {}_{m-k}N_{m-k} \cdot k!\, A_k}{2^{m-k}(bi)^{2(m-k)-1}} + [A_m(a+bi)+B_m] \cdot m!$$

These formulas make it possible to evaluate successively A_0 and B_0, A_1 and B_1, etc. At the same time they not only represent a complete solution of the problem, but provide a practical method of evaluating the numerators of partial fractions, as one can readily see by applying these formulas to particular rational fractions.

The Coefficients ${}_mN_r$. These numbers are easily tabulated using their definition. Arranging them in a table so that ${}_mN_r$ is placed in the mth row and rth column we have:

1						
1	1					
1	3	3				
1	6	15	15			
1	10	45	105	105		
1	15	105	420	945	945	etc.

In practice only the first few rows of the table would be needed.

The numbers ${}_mN_r$ possess many interesting properties, resembling those of the binomial coefficients. A few of these properties are given below.

Summation by Columns. From

$${}_{m-j}N_r = {}_{m-j-1}N_r + (m+r-j-3)_{m-j-1}N_{r-1},$$

letting $j=0,1,\cdots, m-r$, and adding, we get

$${}_mN_r = \sum_{j=0}^{m-r} (m+r-j-3)_{m-j-1}N_{r-1},$$

a formula exhibiting the formation of ${}_mN_r$ from the entries of the $(r-1)$st column of the table.

Summation by Diagonals. From the recurrence formula we deduce readily

$${}_mN_r = {}_{m-1}N_r + \sum_{j=2}^{r} \left[\prod_{i=2}^{i=j} (m+r-2i+1)\right] \cdot {}_{m-j}N_{r+1-j}.$$

Summation of the Entries of any Row. By applying the recurrence relation to each of the entries of the mth row, we get

$$\sum_{n=1}^{m} {}_mN_r = \sum_{r=1}^{m-1} (m+r-1)_{m-1}N_r.$$

The above properties are close analogues of the properties of the binomial coefficients.

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Isolate Numbers

WILLIAM R. RANSOM
Tufts College

The name "imaginary" and the symbol "$\sqrt{-1}$" are algebraic misfortunes. They are unfortunate because they imply and encourage errors. All numbers are creations of the mind and have no material existence, and so are equally *imaginary;* and the well known rule of arithmetic that $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$ does not hold if $a=b=-1$.

The use of the initial "i" from the word imaginary has the effect of maintaining this name against other proposed names, even though the "j" form of this letter has been widely substituted for it. There are excellent reasons for proposing to take the i as the initial of "isolated" and calling i "the isolate number".

Algebra is an extension of arithmetic. It uses the (*signless*) numbers of arithmetic $(0, 1, 2, 3, \cdots)$, the rational numbers derived from them by the four fundamental operations $(+, \times, -, \div)$, and irrationals $(\sqrt{2}, \pi, e,$ etc.), definable from the others by means of limiting processes or Dedekind cuts. Algebra *extends* the number system of arithmetic by the adjunction of three algebraic multipliers, which for the purpose of this note may be called "p", "n", and "i". (p and n are usually called $+1$ and -1). If a signless number is to be measured in a "standard direction" (forward), it is multiplied by p, and if it is to be measured in an opposite direction, it is multiplied by n. It follows that $n(na)=pa$, and $n^2=p$. More generally (not restricting a to be signless), the multiplier p does not change the direction of measurement, the multiplier n does: hence $p\cdot p=p$, $n\cdot n=p$, and $n\cdot p=n$. These are "combinations" referred to in the next paragraph. These multipliers obey the fundamental laws of algebra (commutative, associative, and distributive) and so we have

$$pa+pb+nb=pa+(p+n)b;$$

but since measurements represented by $pa+pb+nb$ mean "a forward, then b forward, then b backward" which is the same as "a forward", it follows that

$$pa+pb+pn=pa+(p+n)b=pa$$

and therefore $p+n=0$. This is another combination.

We say that two numbers "combine" (as in chemistry) if they unite to form something different from the merely *indicated* union.

Thus 2 and 3 combine: they combine to make 5 since $2+3=5$, and they combine to make 6 since $2\times3=6$. If x and y are unknown they may be united as $x+y$ or as $x\cdot y$, but they do not combine, since $x+y$ and xy cannot be otherwise expressed. Using the word *combine* in this sense, we see that p and n combine: we have $nn=p$, $pp=p$, $pn=n$, and $n+p=0$. The number i, however, remains *isolate:* it does not combine with other numbers,—only with itself: $i\cdot i=n$. Thus i will not combine with any other number except itself by multiplication: $2i$, pi, ni, $\cdots$ cannot be expressed in any other way, that is by any single arithmetic or algebraic number. An i will not combine with any other number by addition: in fact if $a+bi=0$, the term bi is zero, and a is zero also. The only combinations which i can form are the self combinations, $i\cdot i=n$, $i\div i=1$, and $i-i=0$, and the combinations $a\cdot 0=0$ and $a\cdot 1=a$, which hold for $a=i$ as a matter of definition of 0 and 1 and not as a special property of i.

Thus i is completely isolate from all other numbers. The name has therefore a justification in the fundamental properties of the number. Moreover this name affords a clue to the dominant procedure to be followed in algebraic work which involves this number. For in such expressions as

$$\frac{1+i}{2-i}, \quad 2i^3(3+1), \quad i\cdot e^{\beta i}, \quad \text{etc.}$$

the first step in dealing with the expressions is to isolate the term in i, so that they can be dealt with in the forms

$$\tfrac{1}{3}+i, \quad -6i+2, \quad i\cdot\cos\beta-\sin\beta, \quad \text{etc.}$$

in which the isolate number appears properly separated from all others, and only once in each expression.

Notes and Comments

Edited by
N. A. COURT

Notes on a Recent Article

The results obtained by Karleton W. Crain ("A Locus Related to the Euler Line", this Magazine, January, 1943, pp. 163-164) may be derived synthetically.

Using the same notation, let A, B be two fixed points on a given circle (O). What is the locus described by the orthocenter R of the triangle PAB when P describes the circle (O)? The answer to this question is readily obtained by considering one of the following properties of the orthocenter R. a. The angles APB, ARB are supplementary. b. The symmetric of R with respect to the base AB lies on the circle (O) (see, for inst., the writer's *College Geometry*, p. 84, Art. 139). c. The circles PAB, RAB are equal (*ibid.*, p. 99, Art. 174). d. The segment PR is equal to twice the distance of the center O of (O) from the line AB (*ibid.*, p. 86, 161).

Thus, using the last of these properties, if O' is the symmetric of O with respect to AB, the segment OO' is equal and parallel to PR, hence $O'R$ is equal and parallel to OP, therefore *the orthocenter R describes the circle (O') which is symmetric to the circle (O) with respect to the line AB.*

Consider a perpendicular to AB which cuts the circle (O) in two points P, P'. The orthocenters R, R' of the triangles PAB, $P'AB$ are the points of intersection R, R' of PP' with the circle (O'), and the radii $O'R$, $O'R'$ are respectively parallel to the radii OP, OP'. From the symmetry of the figure it follows that (O') passes through the points A, B. Let the perpendicular to AB at A meet the circles (O), (O') again in the points P_0, R_0, respectively. Now if the perpendicular $PP'RR'$ moves so that its foot on AB approaches A, one of the points P, P', say P' approaches A, and P approaches P_0, while the point R approaches A and R' approaches R_0.

It may be observed that the construction of the point R of the circle (O) which corresponds to a given position of the point P on (O) can be carried out even if the axis of symmetry AB of the two circles

(O), (O') (which line is also their radical axis) does not cut the circles. We have thus a construction for the orthocenter R of a triangle PAB inscribed in a circle (O), when one vertex P is real and the two other vertices are conjugate imaginary and lie on a given line AB.

Let O_t be any point on the line of centers OO', R any point on (O'), and let the parallel through O_t to the radius $O'R$ meet the line OR in R_t. The circle (O_t) having O for center and O_tR_t for radius is homothetic to (O') with respect to the point O as homothetic center. Hence if two tangents t, t' can be drawn from O to the circle (O'), they will also be tangent to the circle (O_t). As the point O_t varies on OO' we obtain a family of homothetic circles tangent to the same two lines t, t', if these lines are real. N. A. C.

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PROPOSALS

No. 401. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given angle C and side c, construct the scalene triangle ABC so that two external angle bisectors are equal.

Solution by the *Proposer.*

In triangle ABC let the equal external bisectors AD and BE be terminated by BC and CA prolonged. Equating the lengths of AD and BE as found by Stewart's theorem, we have

$$\frac{bc(a-b+c)(a+b-c)}{(c-b)^2}=\frac{ac(-a+b+c)(a+b-c)}{(c-a)^2},$$

which may be simplified, after dropping the factor $c(a-b)(a+b-c)$, to

$$c^3-(a+b)c^2+3abc-ab(a+b)=0. \tag{1}$$

This may be further transformed as follows:

$$-c^2(a+b-c)+ab(4c-c-a-b)=0,$$

$$c^2(s-c)=ab(2c-s)=ab\{c-(s-c)\},$$

$$c^2s(s-c)=ab\{c+(s-c)\}\{c-(s-c)\}=ab\{c^2-(s-c)^2\},$$

$$ab(s-c)^2=c^2\{ab-s(s-c)\}.$$

Since $s(s-c)+(s-a)(s-b)=ab$, we have now

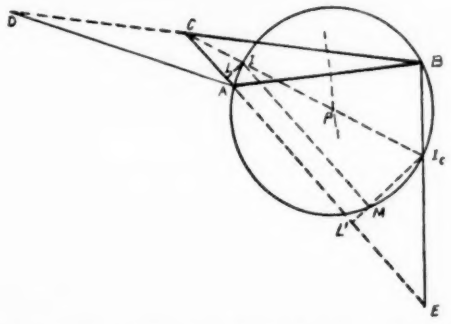
$$\left(\frac{s-c}{c}\right)^2=\frac{s-a}{a}\cdot\frac{s-b}{b}. \tag{2}$$

The familiar relations, $\sin \frac{1}{2}A = \{(s-b)(s-c)/bc\}^{\frac{1}{2}}$, etc., imply $\sin \frac{1}{2}A \cdot \sin \frac{1}{2}B / \sin \frac{1}{2}C = (s-c)/c$ and analogous relations which reduce (2) to

$$\sin^2 \tfrac{1}{2}C = \sin \tfrac{1}{2}A \cdot \sin \tfrac{1}{2}B. \tag{3}$$

This is further reduced by $\sin \frac{1}{2}A = r/AI$ and analogous formulas (where I is center and r is radius of the incircle) to

$$CI^2 = AI \cdot BI. \tag{4}$$



Let P be the intersection of CI with the perpendicular bisector of AB, and let I_C be the excenter on CI. Points A, I, B and I_C lie on circle (P) with center P and radius PA. Draw $r = IL$ and $r_C = I_C L'$ perpendicular to CA, the latter cutting circle (P) at M. Then $IM = LL' = c$ and $\sphericalangle MII_C = \frac{1}{2}\sphericalangle C$. From similar triangles AIL and $I_C IB$, CIL and $II_C M$, we have

$$AI : r = II_C : BI, \qquad r \cdot II_C = AI \cdot BI = CI^2$$

by (4); but

$$CI : r = II_C : r_C - r, \qquad r \cdot II_C = (r_C - r)CI.$$

Hence

$$CI = r_C - r = I_C M.$$

From this analysis the following construction emerges: construct triangle IMI_C, right-angled at M with $IM = c$ and $\sphericalangle MII_C = \frac{1}{2}\sphericalangle C$; prolong $I_C I$ to C with $IC = I_C M$; construct circle (P) having II_C as diameter. The vertices B and A to complete the required triangle are the intersections of the circle (P) with the lines drawn through C and making with CI the angle $\frac{1}{2}C$.

Equation (1) may be written in the following forms:

$$c(c-a)(c-b) + ab(c-a) + ab(c-b) = 0. \tag{1a}$$

$$c^2(c-a) - b(c-a)^2 + ab(c-b) = 0, \tag{1b}$$

$$c(c^2+3ab) = (a+b)(c^2+ab). \tag{1c}$$

Assuming $a>b$, then (1a) shows that $c \not> a$; (1b) shows that $c \not< b$. Now, $(c-a)(c-b)<0$ or $c(a+b)>c^2+ab$. Dividing (1c) by this relation, we have $c^2<a^2+b^2-ab$, which shows that $\not< C<60°$.

EDITOR'S NOTE. In 1840 D. Chr. L. Lehmus (1780-1863) asked J. Steiner (1796-1863) whether a triangle is isosceles if two of its internal bisectors are equal. Steiner answered the question in the affirmative in *Crelle's Journal* (Vol. 28, 1844, p. 375). He also considered the case when two external bisectors are equal and pointed out that in this case the triangle may or may not be isosceles. The scalene triangle with two equal external bisectors has since received a considerable amount of attention. It was named the "pseudo-isosceles" triangle by J. Neuberg (1840-1926) in *Mathesis* (series 2, Vol. 10, 1900, p. 129), where also an article by A. Emmerich may be found in which the proposed problem is considered (p. 131, Art. 5.)

A detailed bibliography on the pseudo-isosceles triangle and a summary of results obtained may be found in *Triangles speciaux*, p. 9, by J. Neuberg, Brussels, 1924, a reprint from *Memoires de la Societe Royale des Sciences de Liege*, series e, Vol. 12. An additional reference: *Mathesis*, 1925, p. 316, note 47, by Neuberg.—N. A. C.

No. 438. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given circle O and point P on the diameter BA prolonged. Construct the secant PDC so that the quadrilateral $ABCD$ is a maximum.

Solution by the *Editors*.

Let the —— points be designated by coordinates $A(-1,0)$, $B(1,0)$, $P(-a,0)$, $C(x_1,y_1)$, $D(x_2,y_2)$; let the equation of the circle be $x^2+y^2=1$ and that of the secant be $y=m(x+a)$. The coordinates of C and D are then obtained by solving these simultaneously. The area of the quadrilateral $ABCD$ is

$$Q=(y_1+y_2+x_1y_2-x_2y_1)/2=am(\sqrt{m^2+1-a^2m^2}+1)/(m^2+1).$$

The familiar necessary condition for a maximum, the vanishing of the derivatives of Q, gives an equation which reduces to

$$(a^2-1)m^2[m^4+(4a^2-2)m^2-3]=0.$$

The positive, real root of m is then found to be

$$m=(2\sqrt{a^4-a^2+1}+1-2a^2)^{\frac{1}{2}}.$$

One method of construction of the required secant with ruler and compass is as follows. Let EF be the diameter perpendicular to PO, and let the line perpendicular to PE at P meet EF at G. Subtract the radius from OG to obtain OH. Then $OG=a^2$, $PH=(a^4-a^2+1)^{\frac{1}{2}}$. On another line mark off LMN with $LM=OF$ and $MN=2(PH-OH)-OF$. If the circle having LN as diameter and the line perpendicular to LN at M intersect at R, the angle MLR is equal to the required inclination of PCD. The proof of the construction is an easy exercise.—E. P. S.

No. 462. Proposed by *Paul D. Thomas*, U. S. Coast and Geodetic Survey.

A variable line turns about the orthocenter of a fixed triangle. Find the locus of the orthopole of this line with respect to the triangle.

Solution by the *Proposer*.

If the vertices of the triangle are $A(-a,b)$, $B(c,b)$, $C\left(0,(b^2-ac)/b\right)$, $b\neq 0$, then the orthocenter is at the origin. Let the equation of the variable line L be $y=mx$. Lines perpendicular to L from A and B respectively meet L in the points

$$P\left(\frac{mb-a}{m^2+1},\frac{m^2b-ma}{m^2+1}\right),\qquad Q\left(\frac{mb+c}{m^2+1},\frac{m^2b+mc}{m^2+1}\right).$$

A line through Q perpendicular to AC is

$$m^2(bx-cy+bc)+m(c^2-b^2)+bx-cy-bc=0. \tag{1}$$

A line through P perpendicular to BC is

$$m^2(bx+ay-ab)+m(a^2-b^2)+bx+ay+ab=0. \tag{2}$$

The intersection of (1) and (2) is the orthopole of L. The elimination of the parameter m between (1) and (2) leads to

$$[b(a-c)x+(b^2-ac)y]^2+4b^4x^2-b^2(b^2-ac)^2=0, \tag{3}$$

the equation of the desired locus, a conic whose center is the orthocenter and which passes through the feet of the altitudes. The discriminant of (3) is $-4b^6(b^2-ac)^4$, and from this it is seen that (3) is nonsingular unless $b^2=ac$, in which case the triangle is right-angled at C and the locus (3) is of course the altitude through C. The discriminant of the second degree terms of (3) is $4b^4(b^2-ac)^2$, which is greater than zero except in the special case just cited. Thus in all other cases (3) is an ellipse.

The locus (3) is a circle if

$$2b(a-c)(b^2-ac)=0,\quad \text{and}\quad b^2(a-c)^2+3b^4=(b^2-ac)^2,$$

whence the only possibility is $a-c=0$ and $b=a/3^{\frac{1}{2}}$. Thus (3) is the circle $3(x^2+y^2)=a^2$, and the vertices of the triangle are $A(-a, a/3^{\frac{1}{2}})$, $B(a, a/3^{\frac{1}{2}})$, $C(0,-2a/3^{\frac{1}{2}})$ which is now equilateral of side $2a$. It is easily shown that this circle is the incircle of the equilateral triangle; that is, the locus of the orthopole of a variable line through the orthocenter of an equilateral triangle is the incircle. This was to be expected but gives a convenient check on the above analysis.

Also solved by *Walter B. Clarke*.

EDITOR'S NOTE. In *Nouvelles Correspondance Mathématique*,* v. 2, 1875, p. 189, J. Neuberg (1840-1926) proposed the following problem (No. 111): Let A', B', C' be the projections of the vertices of a triangle ABC upon a line L. Prove that the lines $A'\alpha$, $B'\beta$, $C'\gamma$ respectively perpendicular to BC, CA, AB meet in a point M. If the line L envelopes a curve of class m, what is the degree of the locus of M? H. Van Aubel offered a solution of the first part of the problem in the same volume, p. 316, and commented, on p. 281, that it is difficult to answer the second part of the question in general, for to a given point M corresponds, in general, more than one line L. Neuberg, replying to his own question in Vol. 4, p. 379 of the same journal, pointed out that to a given point correspond three lines L forming a triangle inscribed in the circumcircle of the given triangle ABC and having for orthocenter the symmetric of M with respect to the orthocenter of ABC. If the line L envelopes a curve of class m, the degree of the locus of M is $2m$. E. Lemoine (1840-1912) seems to have arrived at the answer to the first part of Neuberg's question quite independently in *Journal de Mathématique Élémentaires*, 1884, p. 51.

Such are the early beginnings of the relation between the point M and the line L, named orthopole and orthopolar line by Neuberg in *Mathesis*, 1911, p. 244.

It follows from Neuberg's solution quoted above that if the line L is revolved about a fixed point P (i. e., $m=1$), the locus of M is a curve of degree 2. More precisely it is an ellipse passing through the projections of P upon the three sides of the triangle and having for center the midpoint of the segment joining P to the orthocenter of the triangle.

Consult *Mathesis*, 1896, p. 57 (Soons and Neuberg); 1914, p. 121 and p. 150 (R. Goormaghtigh); 1928, p. 111 (R. Godeau). The *Tohôku Mathematical Journal*, 1926-1927, pp. 77-125 contains a resume of the theory of the orthopole by R. Goormaghtigh. The theory has grown considerably since.—N. A. C.

*Published in Brussels, 1874-1880. under the editorship of E. Catalan (1814-1894)

No. 467. Proposed by *Robert C. Yates*, Louisiana State University.

Find the locus of a point at which equal angles are subtended by two sides of a given triangle ABC.

Note by *Paul D. Thomas*, U. S. Coast and Geodetic Survey.

This problem may be found in the March, 1942 issue of the *American Mathematical Monthly*, pp. 200-201, where the solution is stated by N. A. Court as follows: For the usual meaning of "subtend," the locus consists of part of a strophoid through A and B, with its double point at C, together with the external segment AB. References to detailed solutions and a historical note are also given.

Also solved by the *Proposer*.

PROPOSALS

No. 510. Proposed by *N. A. Court*, University of Oklahoma.

If the radical center of four given spheres coincides with the circumcenter O of the tetrahedron formed by the centers of the given spheres, the tetrahedron formed by the polar planes of O for the given spheres admits O as the center of one of the spheres tangent to the faces of this tetrahedron.

No. 511. Proposed by *Paul D. Thomas*, U. S. Coast and Geodetic Survey.

Find the family of curves for each of which $ds/dA = -2L$, where ds is the element of arc length, dA the element of area, and L the length of the normal from the curve to the Y-axis.

No. 512. Proposed by *V. Thébault*, San Sebastian, Spain.

The locus of the point, the sum of the products of whose distances from the pairs of opposite sides of a regular polygon of $2n$ sides is constant, is a circle concentric with the polygon.

No. 513. Proposed by *E. P. Starke*, Rutgers University.

Assuming that all the roots of the cubic $x^3+ax^2+bx+c=0$ are real, show that the difference between the greatest and the least roots is not less than $(a^2-3b)^{\frac{1}{2}}$ nor greater than $2(a^2-3b)^{\frac{1}{2}}/3^{\frac{1}{2}}$.

No. 514. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Construct a right triangle of given perimeter so that the foot of the altitude on the hypothenuse shall divide that hypothenuse in extreme and mean ratio.

No. 515. Proposed by *N. A. Court*, University of Oklahoma.

Through a point M in the face BCD of the tetrahedron $ABCD$ planes are drawn respectively parallel to the faces ACD, ADB, ABC meeting the edges AB, AC, AD in the points P, Q, R. The lines MP, MQ, MR are produced to the points P', Q', R' so that $MP=2PP'$, $MQ=2QQ'$, $MR=2RR'$. Show that the plane $P'Q'R'$ passes through the vertex A.

The corresponding problem in the plane was considered in the *Journal de Mathématiques Élémentaires*, 1888, pp. 238-239, Q. 256.

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Elementary Mathematics of Artillery Fire. By J. M. Thomas. McGraw-Hill, New York. 256 pages. $2.50.

It is evident that this text was prepared on very short notice to meet a need which was very broadly defined. Under these circumstances the author has succeeded in presenting a comprehensive and coordinated collection of material in a text which may well serve a useful purpose during the present emergency.

The contents fall into three sections. The first four chapters present the geometrical and arithmetical foundations for the subject. A thorough discussion is given in the second chapter of *the accuracy of logarithmic and trigonometric tables* and of *interpolation procedures.* The omission of all angular measure except the natural unit, the radian, and the military unit, the mil, removes the confusion of conversion formulas.

Chapters five through eight are concerned with problems arising in the aiming of artillery. These involve a study of maps, the location and orientation of targets, and the aiming of gun batteries.

The last group of chapters, nine through twelve, presents the mathematics of the actual firing of the gun. The formulas for the trajectory are presented together with the established corrections for air resistance, wind, temperature, muzzle velocity, etc. A valuable discussion of probability as applied to gun fire correction and adjustment is given.

It is believed that the value of this text is very dependent upon the instructor's first hand knowledge of practical problems of artillery fire. A literal use of the text without selection or supplementary direction might be confusing to the average student.

N. A. HALL.

Differential Equations. By Harry W. Reddick. John Wiley and Sons, New York, 1943. ix+245 pages. $2.50.

This text devotes seven chapters to the subject of ordinary differential equations in the following order: definitions with a brief introduction to hyperbolic functions, formation of differential equations, first order differential equations, linear equations with constant coefficients, special higher order equations, simultaneous equations and series solutions.

The student has been kept in mind, as the following facts attest: (1) answers are given to all of the more than 600 problems, (2) several illustrative problems are solved in more than one way, (3) the question of establishing the equivalence of solutions (by use of Jacobians) is given due place, and (4) dimensional units are stressed in physical problems.

It might have been preferable to add $k\pi$, ($k=0, -1, 1$), to arc tan C on page 29. On page 71 the author did not take into account the fact that the left side of (7) is discontinuous on the coordinate axes with the result that the constant difference $C-C'$ has one value in quadrants 1 and 3 and another value in quadrants 2 and 4. On page

36, example 1, *velocity* should be replaced by *speed* in the second sentence. The reviewer would have liked to find a direct attack by inverse D operators in obtaining particular integrals of equations with constant coefficients, especially because of applications to alternating current impedance. However, the above comments are not to be taken as detracting from the generally favorable impression created by the book. It is a pleasure to read a carefully written presentation, which anticipates many of the questions that students commonly raise when so many texts appear as if they had been hastily thrown together in a haphazard fashion.

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Analytic Geometry. By Charles H. Lehmann. John Wiley and Sons, Inc., New York, 1942. 14+424 pp. $3.75.

This book is designed for a five-semester-hour course in plane and solid analytic geometry, based on elementary geometry, plane trigonometry, and some college algebra. Of course, the book can be used for a shorter course. Thirty per cent of the text is devoted to the analytic geometry of three dimensions. Theorems are given in more formal fashion than is customary and remarks follow on special uses or exceptions of the theorems. Great care has been taken in selecting and discussing the illustrative examples and in providing over nineteen hundred exercises which cover the theory of current articles and also anticipate following propositions. Answers are included for a large number of these exercises. Some of the chapters are concluded with a summary in the form of a chart of theorems and formulas. In the appendix is found a set of four tables to assist in calculations.

In plane geometry, the usual preliminary topics are followed by a very good presentation of the discussion of a curve from its equation. After the chapters on the straight line and the circle, it was good to find transformations of coordinates considered before the four chapters on conic sections. To find the equation of the tangent line where the point of contact (x_1,y_1) is given, the slope of the line $y-y_1=m(x-x_1)$ is determined so that the line intersects the conic in coincident points. The method of employing increments h,k for x,y, respectively, could be used or there should be a chapter on the derivative at this point. This part is concluded with adequate treatment of polar coordinates, parametric equations, and higher plane curves.

The five chapters on solid geometry are excellent and are illustrated with very clear diagrams. The treatment of surfaces and curves in space is carefully given and it will furnish a good foundation for a course in calculus.

The printing and page structure is successful except for a few diagrams that are not convenient for reference. The errors are easily detected such as $|OB|=2a\cos\theta$. A wealth of material has been recorded in an exceedingly careful manner.

Oberlin College. MARIE M. JOHNSON.